A Simple Solution for Boundary Layer Flow of Power Law Fluids Past a Semi-infinite Flat Plate

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The steady, two-dimensional, incompressible laminar boundary-layer flow of power law fluids past a semi-infinite flat plate is studied analytically by the method of series expansion and the method of steepest descent (Meksyn's method). The shear stress of the fluids considered is proportional to the nth power of the velocity gradient. The solution obtained is a series of gamma function with the coefficient being a function of the material constant, n. The first ten terms of the series solution are then considered for the sum to determine the velocity gradient at the plate, $f_{\eta\eta}(0)$. The results obtained for $f_{\eta\eta}(0)$ and the coefficient of skin friction are in excellent agreement with the numerical solutions available in the range of $0.1 \leq n \leq 2.5$.

The boundary-layer theory for power law fluids was developed and discussed in detail by Schowalter (7). He showed that for a steady two-dimensional boundary layer flow the shear stress is proportional to the nth power of the velocity gradient and the similar solutions exist for Falkner-Skan types of flow. Later, Hayasi investigated the conditions for the existence of similar solutions for two-dimensional and axisymmetric boundary layers (3). However, the analysis of the fluids flowing past a body was first attempted by Acrivos, Shah, and Petersen (1). They discussed the momentum and heat transfer in boundary-layer flows of power law fluids past external surfaces and solved the flat plate problem by a numerical method as well as by Von Kármán-Pohlhausen integral method. They concluded from the results obtained that the integral method is, in general, less accurate for non-Newtonian fluid flows. In 1965 Berkovskii presented the exact numerical solutions for power law fluids flowing over a flat permeable plate and near the stagnation point (2).

The flat plate problem is solved here theoretically by the method of series expansion and the method of steepest descent (4 to 6). The purposes of this study are to obtain a simple, analytical solution for the flat plate problem, and to show that the powerful method can be used to solve the boundary layer problems of power law fluids.

GOVERNING EQUATIONS

For a steady, two-dimensional, incompressible, laminar flow past a semi-infinite flat plate, the governing boundary layer equations of continuity and motion are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\rho}\,\frac{\partial \tau_{xy}}{\partial y} \tag{2}$$

respectively. The equation of state for the boundary layer fluid flow is (7)

$$\tau_{xy} = K\left(\frac{\partial u}{\partial y}\right)^n \tag{3}$$

The boundary conditions to be satisfied in this problem are

$$\text{at } y = 0 : \quad u = 0, \quad v = 0 \tag{4}$$

$$\text{as } y \to \infty : \quad u \to U = \text{constant} \tag{5}$$

It is rather well known that a similarity transformation exists for the problem considered. A stream function $\psi(x, y)$ can be introduced from the continuity equation such that

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \tag{6}$$

Letting (see Appendix)

$$\eta = yU\left[\frac{K}{\rho}\,n(n+1)\,U^{2n-1}x\right]^{-\frac{1}{n+1}}, \psi(x, y) = \left[\frac{K}{\rho}\,n(n+1)\,U^{2n-1}x\right]^{\frac{1}{n+1}} f(\eta) \tag{7}$$

the boundary layer equation of motion, Equation (2), with the expression for shear stress in Equation (3) takes the form

$$f_{\eta\eta\eta} + f f_{\eta\eta}{}^{2-n} = 0 \tag{8}$$

where the subscript η denotes the differentiation. The boundary conditions, Equations (4) and (5), now become

$$f(0) = 0, \quad f_\eta(0) = 0 \tag{9}$$

$$f_\eta(\eta \to \infty) \to 1 \tag{10}$$

By use of the transformations, Equation (7), the shear stress defined in Equation (3) can be written as

$$\tau_{xy} = \rho U^2\,[n(n+1)]^{-\frac{n}{n+1}}\, f^n{}_{\eta\eta}\, N_{Re_x}{}^{-\frac{1}{n+1}} \tag{11}$$

where $N_{Re_x} \equiv (\rho U^{2-n}\, x^n)/K$ is the Reynolds number based on x. The coefficient of skin friction is then defined as

$$C_d = [n(n+1)]^{-\frac{n}{n+1}}\, f^n{}_{\eta\eta}(0) \tag{12}$$

It is noted that the definition for C_d is exactly the same as $C(n)$ of Acrivos, et al. (1).

METHOD OF SOLUTION

To solve the nonlinear ordinary differential equation, Equation (8), we first expand the function $f(\eta)$ in a power series of η such that

$$f(\eta) = \sum_{m=0}^{\infty} \frac{a_m}{m!}\,\eta^m \tag{13}$$

Application of the boundary conditions, Equation (9), and substitution of Equation (13) into Equation (8) yield

$$a_0 = 0,\ a_1 = 0,\ a_2 = a,\ a_3 = 0,\ a_4 = 0,\ a_5 = -a^{2+k},$$
$$a_6 = 0,\ a_7 = 0,\ a_8 = (11 + 10k)\ a^{3+2k},\ a_9 = 0,$$
$$a_{10} = 0,\ a_{11} = -(375 + 934k + 560k^2)\ a^{4+3k},$$
$$a_{12} = 0,\ a_{13} = 0,\ a_{14} = (27{,}897 + 126{,}136k$$
$$+ 190{,}640k^2 + 92{,}400k^3)\ a^{5+4k},\ a_{15} = 0,\ \text{etc.,} \quad (14)$$

where $k \equiv 1 - n$. It is noted that all the expansion coefficients a_m found are expressed in terms of the unknown coefficient a_2 ($\equiv a$) and n and can be reduced to those for Blasius problem if we set $k = 0$ (that is $n = 1$). Now, by use of the coefficients found and Equation (13) we have

$$ff^k_{\eta\eta} = \sum_{m=0}^{\infty} d_m\ \eta^m = \frac{a^{1+k}}{2!}\eta^2$$
$$- \frac{(1 + 10k)\ a^{2+2k}}{5!}\eta^5 + \frac{(11 + 94k + 560k^2)\ a^{3+3k}}{8!}\eta^8$$
$$- \frac{(375 + 3{,}816k + 21{,}240k^2 + 92{,}400k^3)\ a^{4+4k}}{11!}\eta^{11}$$
$$+ --- \quad (15)$$

Equation (15) is needed next in determining the unknown coefficient $a_2 \equiv a$ by the method of steepest descent.

To determine a_2 we then integrate Equation (8) twice and find that

$$f_\eta(\eta) = f_{\eta\eta}(0) \int_0^\eta e^{-F(\eta)}\, d\eta \quad (16)$$

in which

$$F(\eta) = \int_0^\eta ff^k_{\eta\eta}\, d\eta \quad (17)$$

Integrating Equation (17) by use of Equation (15) and letting

$$F(\eta) = \eta^3 \sum_{m=0}^{\infty} c_m\ \eta^m \equiv \tau \quad (18)$$

in which the coefficients c_m are

$$c_m = \frac{d_{m+2}}{m+3} \quad (19)$$

with d_m defined in Equation (15) we find by the inversion theorem that

$$\eta = \sum_{m=0}^{\infty} \frac{b_m}{m+1}\ \tau^{\frac{1}{3}(m+1)} \quad (20)$$

The coefficient b_m in Equation (20) can easily be shown to be the coefficient of η^m in the expression (5, 6)

$$[c_0 + c_1\ \eta + c_2\ \eta^2 + c_3\ \eta^3 + c_4\ \eta^4 + ---]^{-\frac{1}{3}(m+1)} \quad (21)$$

They are found to be (4)

$$b_0 = \left(\frac{a^{1+k}}{6}\right)^{-\frac{1}{3}},\ b_1 = 0,\ b_2 = 0,$$
$$b_3 = \frac{(1 + 10k)}{15}\ b_0,\ b_4 = 0,\ b_5 = 0,$$
$$b_6 = -\frac{(2 - 23k - 70k^2)}{360}\ b_0,\ b_7 = 0,\ b_8 = 0$$
$$b_9 = \frac{(322 - 6{,}405k + 35{,}175k^2 + 38{,}500k^3)}{124{,}740}\ b_0,\ \text{etc.} \quad (22)$$

The integral in Equation (16) can now be written in terms of the variable τ and integrated, that is

$$f_\eta(\eta) = f_{\eta\eta}(0) \int_0^\tau \frac{1}{3} e^{-\tau} \sum_{m=0}^{\infty} b_m\ \tau^{\frac{1}{3}(m-2)}\, d\tau = a_2 \sum_{m=0}^{\infty} b_m\ \Gamma_\tau\left(\frac{m+1}{3}\right) \quad (23)$$

where Γ_τ is an incomplete gamma function. Once the value for $f_{\eta\eta}(0) \equiv a_2$ is determined by the boundary condition at infinity, Equation (23) can give us the velocity profile. The corresponding value of η for a given value of τ in Equation (23) can be obtained from Equation (20).

Application of the boundary condition at infinity, Equation (10), to Equation (23) gives

$$1 = \frac{1}{3} a_2 \sum_{m=0}^{\infty} b_m\ \Gamma\left(\frac{m+1}{3}\right) \quad (24)$$

This is the relation to be used for determining the only unknown coefficient $f_{\eta\eta}(0) \equiv a_2$. Now if we consider the first ten terms of the series in Equation (24) for evaluating a_2, we find after some manipulations that

$$a_2 = \left[\frac{0.61628}{S}\right]^{\frac{3}{2-k}} \quad (25)$$

with $k \equiv 1 - n$ and

$$S = 1 + \frac{(1 + 10k)}{45} - \frac{(2 - 23k - 70k^2)}{810}$$
$$+ \frac{(322 - 6{,}405k + 35{,}175k^2 + 38{,}500k^3)}{1{,}202{,}850} \quad (26)$$

For a given value of n ($k \equiv 1 - n$), a_2 can be calculated rather easily from Equations (25) and (26).

It is noted here that when the value of n is zero, Equation (3) indicates that the shear stress is constant everywhere in the flow field and the governing equation will certainly be different from the one we have considered in this study. Thus, the solution obtained is not valid for the case when the value of k equals to one (that is $n = 0$). Fortunately, the usual range for the value of k in practice lies between -1 and 0.9. Mathematically, if the series in Equation (24) [or terms in Equation (26)] converges rather slowly or even becomes divergent, application of Euler's transformation (6) are necessary to find the sum of the series.

RESULT AND DISCUSSION

The first ten terms of the series in Equation (24) are used without any application of Euler's transformation to find the sum for evaluating the velocity gradient at the plate, $f_{\eta\eta}(0)$. This means that terms in Equation (26) converge nicely and that $f_{\eta\eta}(0) \equiv a_2$ presented here is calculated directly from Equation (25) for a given value of n.

The value of $f_{\eta\eta}(0)$ obtained for pseudoplastic fluids is given in Table 1 for the range of $0.1 \leq n \leq 1$. In the course of calculation we found that the convergence of the term for S defined in Equation (26) is generally slowing down as the value of n decreases. (This is because when $n = 0$, the governing equation is different from the one we have considered.) This indicates the accuracy of $f_{\eta\eta}(0)$ calculated from Equation (25) decreases as the degree of pseudoplasticity increases. To make sure the results obtained are accurate and acceptable, they are compared with the exact numerical solutions of Berkovskii which are also presented in Table 1. It shows that the difference in $f_{\eta\eta}(0)$ between the two works is less than 0.5% when $0.5 \leq n \leq 1$, and increases to 6.6% when $n = 0.1$. This is expected because only the first ten terms of the series in Equation (24) are used for sum without any application of Euler's transformation. Based on the results obtained for $f_{\eta\eta}(0)$ the coefficient of skin friction defined in Equation (12) is also calculated and given in Table 1. A comparison of C_d between the two works again shows that the difference in C_d generally increases as the degree of pseudoplasticity increases. However, the maximum difference is now less than 0.8% for $0.1 \leq n \leq 1.0$. This is no surprise at all, since C_d is proportional to the nth power of $f_{\eta\eta}(0)$.

TABLE 1. VALUES OF $f_{\eta\eta}(0)$ AND C_d FOR PSEUDOPLASTIC FLUIDS

	Equation 25		Berkovskii	
η	$f_{\eta\eta}(0)$	C_d	$f_{\eta\eta}(0)$	C_d*
0.1	0.1160	0.9853	0.1088	0.9791
0.2	0.1528	0.8712	0.1472	0.8647
0.3	0.1922	0.7577	0.1883	0.7531
0.4	0.2332	0.6592	0.2306	0.6563
0.5	0.2748	0.5770	0.2735	0.5756
0.6	0.3163	0.5090	0.3157	0.5084
0.7	0.3569	0.4526	0.3567	0.4524
0.8	0.3963	0.4055	0.3962	0.4055
0.9	0.4339	0.3658	0.4339	0.3658
1.0	0.4696	0.3321	0.4696	0.3321

* Calculated by the present author.

Table 2 presents the velocity gradient at the plate and the coefficient of skin friction for dilatant fluids. According to the values calculated for the terms in Equation (26) we found that the accuracy of $f_{\eta\eta}(0)$ obtained for $1.0 \leq n \leq 2.5$ is in the order of magnitude of that when $n = 0.2$. It is therefore believed that the results given in Table 2 are acceptable in the range $1.0 \leq n \leq 2.5$. Furthermore, the coefficient of skin friction obtained for $n =$ 1.5, 2.0, 2.5, and 3.0 agrees quite well with numerical results of Acrivos, et al. (*1*).

TABLE 2. VALUES OF $f_{\eta\eta}(0)$ AND C_d FOR DILATANT FLUIDS

n	$f_{\eta\eta}(0)$	C_d	n	$f_{\eta\eta}(0)$	C_d
1.1	0.5033	0.3030	1.9	0.7003	0.1661
1.2	0.5347	0.2778	2.0	0.7180	0.1561
1.3	0.5640	0.2558	2.1	0.7346	0.1471
1.4	0.5912	0.2363	2.2	0.7506	0.1391
1.5	0.6164	0.2190	2.3	0.7660	0.1319
1.6	0.6398	0.2036	2.4	0.7810	0.1256
1.7	0.6614	0.1897	2.5	0.7959	0.1200
1.8	0.6815	0.1773	3.0	0.8753	0.1040

We can finally conclude from the above discussions and comparisons that the simple solution for $f_{\eta\eta}(0)$, Equation (25), obtained by series expansion and the method of steepest descent can give us the skin friction as accurate as that obtained by other methods. It is, furthermore, believed that Falkner-Skan flows of power law fluids can be solved analytically by the same method with some application of Euler's transformation (5).

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NOTATION

- C_d = coefficient of skin friction defined by Equation (12)
- $f(\eta)$ = dimensionless stream function
- k = $1 - n$
- K, n = parameters in power law model, Equation (3)
- N_{Re_x} = $(\rho U^{2-n} x^n)/K$, Reynolds number based on x
- u = velocity component in x direction
- U = external velocity
- v = velocity component in y direction
- x = distance along the plate from the leading edge
- y = distance normal to the plate

Greek Letters

- Γ = gamma function
- Γ_τ = incomplete gamma function
- η = stretched dimensionless ordinate defined by Equation (7)
- ρ = density
- τ = variable defined by Equation (18)
- τ_{xy} = the shear stress
- ψ = stream function

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APPENDIX: SIMILARITY TRANSFORMATION

By introducing a stream function $\psi(x, y)$, Equation (6), the boundary layer equation, Equation (2), with the equation of state defined by Equation (3) can be written as

$$\psi_y \, \psi_{yx} - \psi_x \, \psi_{yy} = \frac{K}{\rho} n \, \psi_{yy}{}^{n-1} \, \psi_{yyy} \tag{A1}$$

in which the subscripts x and y denote the differentiation respect to x and y. The boundary conditions of the problem now become

$$\psi(x, 0) = 0, \qquad \psi_y(x, 0) = 0 \tag{A2}$$

$$\psi_y(x, y \to \infty) \to U = \text{constant} \tag{A3}$$

To find a similar solution for the problem, Equations (A1), (A2) and (A3), two scale factors, $a(x)$ and $b(x)$, both functions of x are introduced such that

$$\eta = \frac{y}{a(x)}, \; f(\eta) = \frac{\psi(x, y)}{b(x)} \tag{A4}$$

The two scale factors, $a(x)$ and $b(x)$ are to be found next. In order not to complicate the boundary conditions of the transformed problem, Equation (A3) indicates that

$$a(x) = b(x)/U \tag{A5}$$

By use of the transformations, Equation (A4), and the relation in Equation (A5), Equation (A1) becomes

$$-\frac{U^2}{b(x)}\frac{db(x)}{dx}ff_{\eta\eta}=\frac{K}{\rho}n\frac{U^{2n+1}}{b(x)^{n+1}}f_{\eta\eta}{}^{n-1}f_{\eta\eta\eta} \quad (A6)$$

Equation (A6) indicates the existence of a similar solution if

$$\frac{U^2}{b(x)}\frac{db(x)}{dx}=C_1\frac{K}{\rho}n\frac{U^{2n+1}}{b(x)^{n+1}} \quad (A7)$$

in which C_1 is an arbitrary constant. Rewriting Equation (A7) in the form

$$b^n(x)\frac{db(x)}{dx}=C_1\frac{K}{\rho}n\,U^{2n-1}$$

and integrating once yields

$$b(x)=\left[C_1\frac{K}{\rho}n(n+1)\,U^{2n-1}x\right]^{\frac{1}{n+1}} \quad (A8)$$

For the case when $x \neq 0$, the substitution of Equation (A8) into Equation (A6) yields

$$\frac{1}{C_1}f_{\eta\eta\eta}f_{\eta\eta}{}^{n-1}+ff_{\eta\eta}=0 \quad (A9)$$

Furthermore, this equation can be written as

$$f_{\eta\eta\eta}+C_1 ff_{\eta\eta}{}^{2-n}=0 \quad (A10)$$

Without loss of generality the arbitrary constant, C_1, can be set equal to one. Then, the substitution of Equation (A8) into Equation (A4) with the relation, Equation (A5), gives the similarity transformation, Equation (7), and Equation (A10) becomes Equation (8). Similarly, the transformed boundary conditions for Equations (A2) and (A3) are those given in Equations (9) and (10).

The Experimental Determination of the Volumetric Properties and Virial Coefficients of the Methane-Ethylene System

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An apparatus was constructed for making isochoric measurements of P-V-T properties. Experimental compressibility factors were obtained and are presented for methane, ethylene, and four intermediate mixtures at 60, 40, and 20°F., with pressures from 260 to 2,220 lb./sq.in.abs. Second and third virial coefficients and interaction virial coefficients were determined from the data. The compressibility factors and the virial coefficients are compared with the Benedict-Webb-Rubin equation of state, applied in its original form and in a recent generalized form.

Volumetric (or compressibility factor) data are of industrial interest in process design calculations. The data are also of value in the calculation of derived thermodynamic quantities and in the further development of generalized methods for estimating thermodynamic properties from a minimum of direct data.

In the past, several experimental facilities have been developed for determining compressibility factors of mixtures or pure components along isochoric (constant density) paths. The apparatus of Goodwin (*7*), Michels (*16*, *20*), and Solbrig and Ellington (*28*) are of the isochoric type. In principle the compressibility factors for a constant composition system become established with the simultaneous knowledge of the three quantities pressure, temperature, and density. At selected points along an isochoric path the pressure and temperature are directly measured, and the data thus consist of a series of *P-T* points at constant density. The value of the density may be determined by separate measurement of the mass and volume quantities or, alternately, the density may be determined from the previously measured compressibility factors of the sample at a reference temperature.

In the apparatus described herein, the density of the sample is determined from the compressibility factors along the reference isotherm (77°F.). The isotherm is independently determined using a Burnett type of apparatus (*3*).

The methane-ethylene binary system was selected for this study. The compressibility factors of the methane system have been extensively reported from −274 to 650°F. and at pressures as high as 1,000 atm. (*4, 10, 13, 19, 21* to *23, 27, 30*). By far the largest portion of the experimental data is reported above the critical temperature

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